Next we give Tsai's write-up.

Let $x \in \mathbb{Q}$ and let n be a positive integer. We prove by induction on n that

$$f(nx) \;=\; nf(x) + 2547(n-1)\,.$$

For $n = 1$, this is trivial. Assume the above equation holds for some positive integer n and for all $x \in \mathbb{Q}$. The induction step is completed by the calculation

$$\begin{aligned} f\big((n+1)x\big) &= f(nx+x) \;=\; f(nx) + f(x) + 2547 \\ &= nf(x) + 2547(n-1) + f(x) + 2547 \\ &= (n+1)f(x) + 2547n\,. \end{aligned}$$

Now, we have $f(2004 \cdot 2547) = 2004f(2547) + 2547 \cdot 2003$ and also $f(2547 \cdot 2004) = 2547f(2004) + 2547 \cdot 2546$; thus,

$$\begin{aligned} f(2547) &= \frac{2547f(2004) + 2546 \cdot 2547 - 2547 \cdot 2003}{2004} \\ &= \frac{2547^2 + 2546 \cdot 2547 - 2547 \cdot 2003}{2004} \;=\; \frac{1311705}{334}\,. \end{aligned}$$

Remark. Let $C \in \mathbb{Q}$ and let $f : \mathbb{Q} \to \mathbb{Q}$. Then $f(x+y) = f(x) + f(y) + C$ for all x, $y \in \mathbb{Q}$ if and only if $f(nx) = nf(x) + C(n-1)$ for all $x \in \mathbb{Q}$ and all $n \in \mathbb{N}$.

3. Let a, b, and c be positive real numbers such that $a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $a^3 + b^3 + c^3 \geq a + b + c$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Vedula N. Murty, Dover, PA, USA; George Tsapakidis, Agrinio, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give Alt's generalization.

By Jensen's Inequality, we have

$$\frac{a^3 + b^3 + c^3}{3} \;\geq\; \left(\frac{a+b+c}{3}\right)^3.$$

We also have

$$(a+b+c)^2 \;\geq\; (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \;\geq\; 9\,.$$

We conclude that

$$a^3 + b^3 + c^3 \;\geq\; \frac{(a+b+c)(a+b+c)^2}{9} \;\geq\; a+b+c\,.$$

Generalization. Let n be a non-negative integer. With the same hypotheses, we have

$$a^{n+1}+b^{n+1}+c^{n+1} \;\geq\; a^{n-1}+b^{n-1}+c^{n-1}\,.$$

Proof. For non-negative integers n and m, we have

$$a^{n+m}+b^{n+m}+c^{n+m} \;\geq\; \frac{(a^n+b^n+c^n)(a^m+b^m+c^m)}{3}\,.$$

Indeed,

$$\begin{aligned} &3(a^{n+m}+b^{n+m}+c^{n+m})-(a^n+b^n+c^n)(a^m+b^m+c^m) \\ &\quad= \sum_{\text{cyclic}}(a^{n+m}+b^{n+m}-a^nb^m-a^mb^n) \\ &\quad= \sum_{\text{cyclic}}(a^n-b^n)(a^m-b^m) \;\geq\; 0\,. \end{aligned}$$

Using this inequality and

$$a^2+b^2+c^2 \;\geq\; \frac{(a+b+c)^2}{3} \;\geq\; \frac{(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})}{3} \;\geq\; \frac{9}{3} \;=\; 3\,,$$

we immediately obtain

$$\begin{aligned} a^{n+1}+b^{n+1}+c^{n+1} &\geq \left(a^{n-1}+b^{n-1}+c^{n-1}\right)\left(\frac{a^2+b^2+c^2}{3}\right) \\ &\geq a^{n-1}+b^{n-1}+c^{n-1}\,. \end{aligned}$$

6. Let $ABCD$ be a convex quadrilateral. Prove that

$$[ABCD] \;\leq\; \tfrac{1}{4}\,(AB^2+BC^2+CD^2+DA^2)\,.$$

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the solution by Zvonaru.

We have

$$\begin{aligned} [ABCD] &= [ABC] \;+\; [CDA] \\ &= \tfrac{1}{2}\cdot AB\cdot BC\cdot\sin\angle ABC \;+\; \tfrac{1}{2}CD\cdot DA\cdot\sin\angle CDA \\ &\leq \tfrac{1}{2}AB\cdot BC \;+\; \tfrac{1}{2}CD\cdot DA \\ &\leq \tfrac{1}{2}\left(\tfrac{1}{2}(AB^2+BC^2)+\tfrac{1}{2}(CD^2+DA^2)\right) \\ &= \tfrac{1}{4}(AB^2+BC^2+CD^2+DA^2)\,. \end{aligned}$$

Equality holds if and only if $ABCD$ is a square.